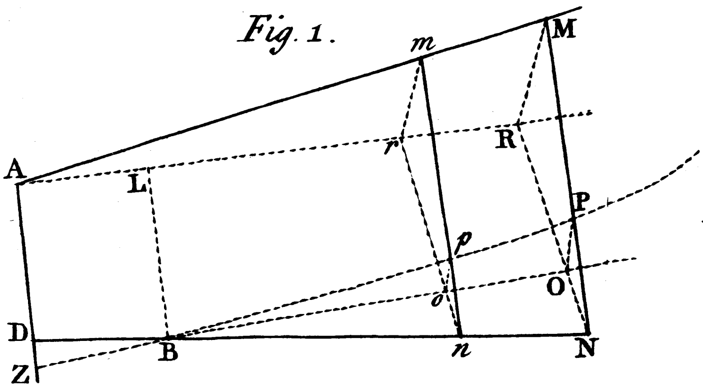
XXXIX. *A Letter to the Right Honourable* Hugh *Earl of* Marchmont, *F. R. S. concerning the Sections of a Solid, hitherto not considered by Geometers; from* William Brakenridge, *D. D. Rector of* St. Michael Bassishaw London, *and F. R. S.*

My Lord,

Read Dec. 20, 1759.

YOUR knowlege in Geometry, and the other Sciences that depend upon it, makes me presume to lay before you the following speculations. Your benevolence to all philosophical Inquiries encourages me, and the personal regard I have for your Lordship induces me to do myself this honour; and tho' what I offer at present may be of no great consequence, I am persuaded, that every little accession to our knowlege will give you some pleasure, as you very well know, that all our improvements in science are slow, and from small beginnings. You have here a new method of considering some geometrical curves, from the sections of a solid, hitherto not taken notice of, and by which, in particular, you will see, that the two infinite curve lines from the section of the cone, are also the sections of this; which may be of some use, as it seems to extend our views of their nature and properties. The description of it is very easy and obvious, and it has something remarkable in its form, that tho', in the most simple case, it is generated and bounded by right lines, the surface is incurvated. The solid is thus described.

Fig. 1.



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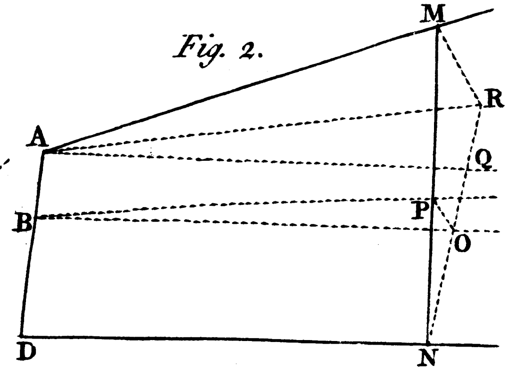
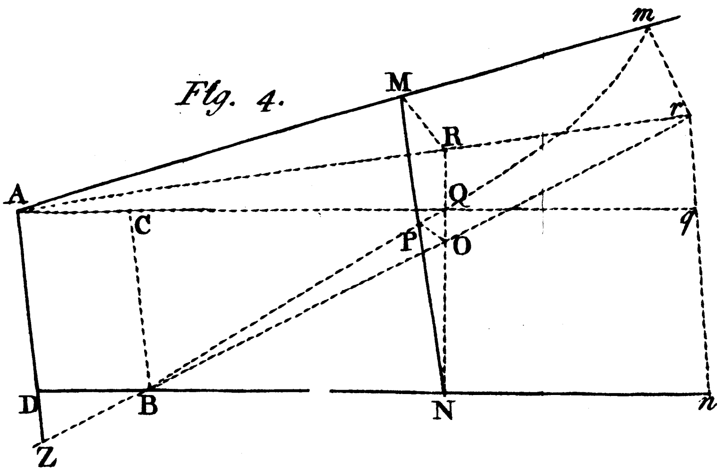


Fig. 4.



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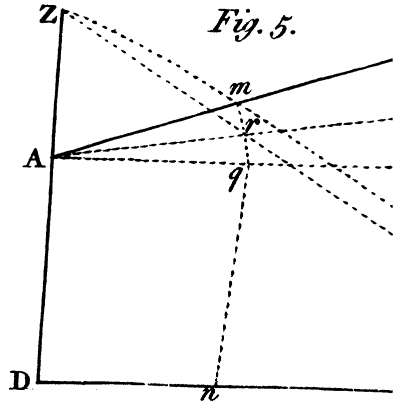

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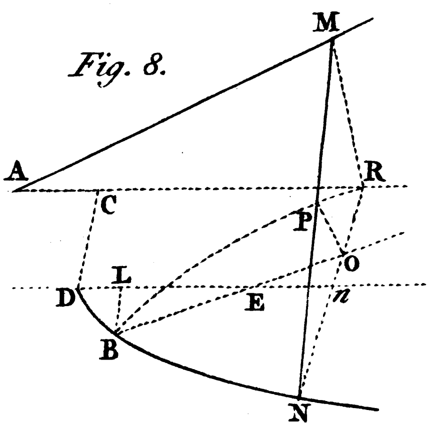


Fig. 9.

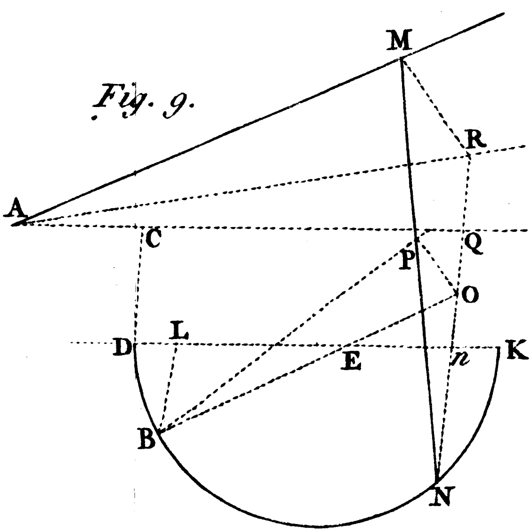
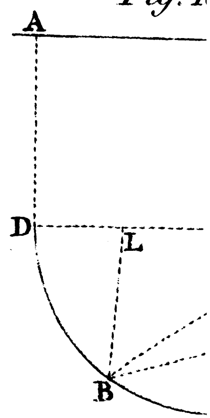


Fig. 1



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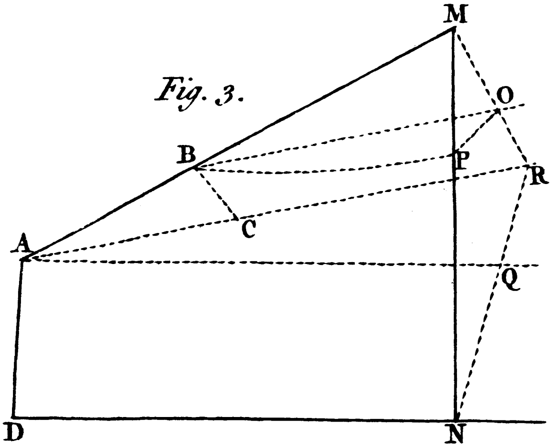

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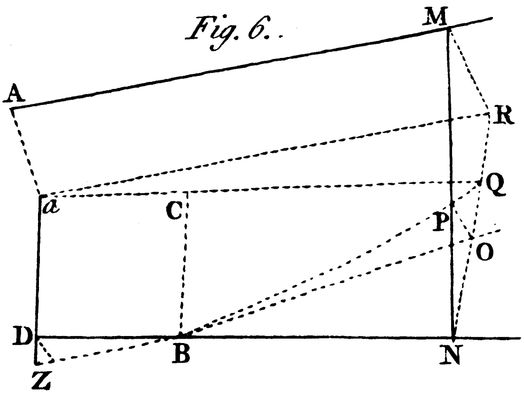

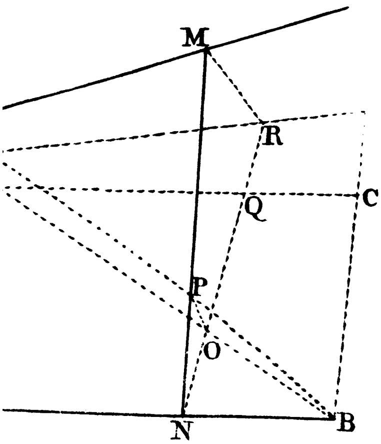

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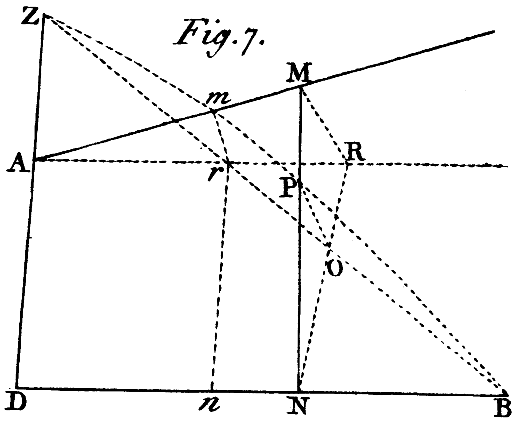


Fig. 10.

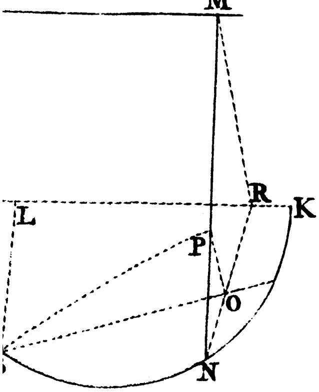
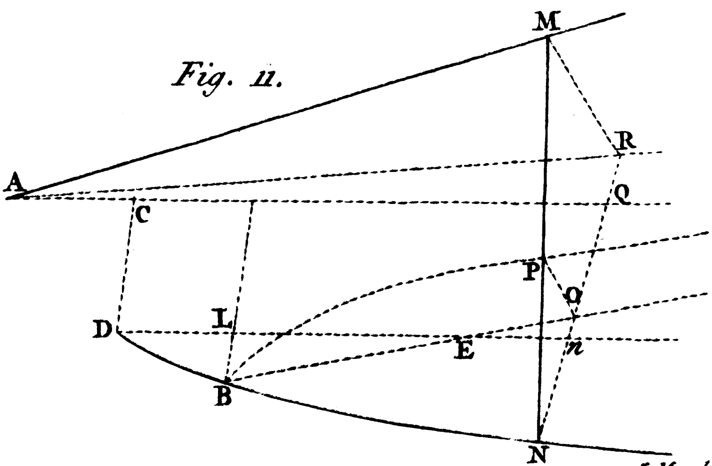


Fig. 11.



J. Mynde sc.

Let D N be a right line drawn on a plane, and a point given A at any diſtance from the line, from which having raiſed, above the plane, the right line A M in any given angle, and drawing from the ſame point A the line A D, meeting the line D N on the plane in D, make another plane M N R to move parallel to itſelf, and to that line A D in a given angle to the firſt plane; and then, ſuppoſe the interſections of it M, N, *m*, *n*, &c. with the lines A M and D N, to be continually joined with right lines M N, *m n*, &c. and there will be generated an incurvated ſurface by them, and bounded by the lines A M, M N, N D, A D. *Plate* X. *Fig.* 1.

The ſame ſurface will be deſcribed, if a line M N be ſuppoſed to move in ſuch manner, along the lines A M and D N, that the parts A M and D N be continually in a given ratio. For let a plane be drawn thro' the raiſed line A M, perpendicular to the given plane A D N, and their common interſection be the line A R; then having joined the points A and D, from the point N, while the line M N is moving, let there be continually drawn a line N R parallel to D A, meeting the line of interſection A R of the planes in R, and the proportion of D N to A R will always be given. But as the ratio of D N to A M is given, the ratio of A M to A R will alſo be given; and the line R M being drawn, the angle A R M will be given; and therefore the plane of the triangle M R N will move parallel to itſelf, and to the line A D, making the given angle M R A with the plane A D N, and the line M N will generate the ſame ſurface as before.

It

It is evident this incurvated ſurface may be infinitely extended on all ſides beyond the lines A M, M N, N D, A D, and as well below the given plane A D N as above it; and therefore the various ſections of it, if continued, will be infinite lines.

The line A M, raiſed above the given plane A D N, may be called the vertical Directrix.

And if thro' this directrix A M there be drawn a plane A M R in any given angle to the plane A D N, interſecting the moving plane N M R in the line M R, and meeting the given plane A D N in R; then the points A, R, and R, N, being joined, there will be a trapezium A R N D, formed, that may be named the Baſe, and in which the line D N may be called the Directrix of the Baſe.

And the plane A M R paſſing thro' the vertical directrix and the moving plane M N R, together with the baſe A R N D, and the curve ſurface, will make a ſolid A M R N D A, ſomething in the form of a wedge.

In this ſolid there may be made five ſections in a given angle with the baſe, or parallel to it; viz, one parallel to the moved plane; one parallel to the directrix of the baſe; another parallel to the vertical directrix; a fourth parallel to the baſe; and a fifth interſecting both the directrices. And in all theſe caſes, when the directrices are right lines, the ſections will be either the conic hyperbola, or the parabola, or right lines.

1ſt, If the ſolid A M R N D A be cut by a plane parallel to the moved or generating plane N M R, or the line A D, the ſection will be a right line. This is evident from the deſcription, *Fig.* 1.

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2. If the section be made by a plane B P O parallel to the directrix of the base D N, and in any given angle to the base, it will be an hyperbola, *Fig* 2.

Let the section P B O meet the base in the line B O, which will be parallel to D N, and make the moving plane N M R intersect the base in N R, the vertical directrix in M, and the section in P O; by which D B=N O. From the point M draw M R parallel to P O, and imagine a plane to pass through the directrix A M and the line M R, meeting the base in A R; which will be given in position. Then thro' the point A draw A Q, parallel to D N, intersecting N R in Q, and making N Q=A D; and the two triangles A R Q, A M R, will have all their angles given, and the proportion of their sides. And therefore the ratio of A Q to Q R, and of A Q to M R, will be given. Make A D=a, B D=b, A Q : Q R :: $a:q$, and A Q : M R :: $a:m$, the abscisse B O=x, and the ordinate P O=y; from which Q R=$\frac{qx}{a}$, N R=$a+\frac{qx}{a}$, M R=$\frac{mx}{a}$. And then from the similar triangles N O P, N M R, the analogy N O : O P :: N R : M R will give the equation $yxq+ya^2=xbm$, and the curve B P is an hyperbola.

3. If the section P B be made by a plane parallel to the vertical directrix A M, it will be an hyperbola. *Fig*. 1.

Let the moving plane M R N, parallel to D A, intersect the base in R N, and the vertical directrix in M, and make the plane of the section B P O cut the moving plane N M R in P O, and the base in the line B O, meeting D A in Z, and the base directrix D N in B; from the point M draw the line

M R parallel to the plane of the ſection, and meeting the baſe in R; and the lines M R and P O will be parallel. Then thro' the vertical A M, and the line M R, make a plane to paſs, which will be parallel to the ſection, interſecting the baſe in A R, and the lines A R and B O will alſo be parallel. Draw B L parallel to D A, or N R, paſſing A R in L, and then B O=L R, B L=R O=A Z, and A L=B Z. And the line A R being given in poſition, the lines B L and A L will be given. The angles alſo M A R and N B O being given, the ratio of A R to R M, and of B O to O N, will be given. Suppoſe A D z $=$B L=a, A L=c the abſciſſe B O=x, the ordinate P O=y, and A R : R M : : a : m, and B O : O N : : a : n, from which we have O N=$\frac{nx}{a}$, R M=$\frac{mc+mx}{a}$; N R=$a+\frac{nx}{a}$. And then, in the triangles on the moving plane, which are ſimilar, M R N, P O N, the analogy N O : P O : : N R : R M will produce the equation $y\,a^3+n\,x\,a\,y=x^2n\,m+x\,n\,m\,c$; which ſhews the curve B P to be an hyperbola; and the figure is convex to the baſe.

4. The ſection being made parallel to the baſe, it will be the ſame curve. *Fig.* 3.

Having thro' the vertical directrix made a plane A M R to paſs perpendicular to the baſe, let the ſection B P O, and the baſe parallel to it, meet that plane in the lines B O and A R, and alſo the moving plane N M R in the lines P O and N R. From the point A draw A Q parallel to the directrix D N of the baſe, and A D parallel to N R, and from B the line B C parallel to O R, meeting A R in C; and then

then the lines A D, A C, and B C, will be given, and the angles R A Q, M A R, being alſo given, the proportion of A R to Q R, and of B O to M O, will be known. Make A D $=a$, A C $=c$, the abſciſſe B O $=x$, the ordinate P O $=y$, and A R : R Q :: $a : q$; from which we have M O : M R :: $x : x + c$, Q R $= \frac{cq + qx}{a}$. And becauſe, in the ſimilar triangles M P O, and M N R, in the moving plane, we have M O : P O :: M R : R N, there will reſult the equation $yxa + yca = x^2q + xcq + xa^2$; which denotes the curve B P to be an hyperbola.

5. If the ſection is made ſo as to meet the two directrices, the curve will be alſo an hyperbola. *Fig.* 4.

Let the ſection B P O meet the directrices in B, m, and interſect the plane of the baſe in the line B O; and make the moving plane N M R to be cut by the ſection in P O, and to meet the vertical directrix in M. Then from the point M draw M R parallel to P O; and thro' the lines A M, M R, imagine, as before, a plane to paſs, interſecting the baſe in the line A R r, and meeting the line B O in r, and the ſection in $m\,r$. And from A draw A D parallel to N R, and A Q to D N; and thro' r make $n\,r$ parallel to N R, and to meet A Q in q, and D N in n. Draw alſo from the point B the line B C parallel to A D, and meeting A Q in C. The lines then A q, A r, $m\,r$, $r\,q$, B r, B n, $r\,n$, will be given. Make therefore AD$=a=$BC, AQ : QR :: $a : q$, and AQ : RM :: $a : m$, B r : B n :: $a : b$, B r : $r\,n$ :: $a : n$, A C$=$D B $=c$; the abſciſſe B O$=x$; the ordinate O P$=y$. From which we have B N$=\frac{bx}{a}$, O N$=\frac{nx}{a}$, A Q$=c+$

bx

$\frac{bx}{a}$, RQ$=\frac{qca+qbx}{a^2}$, MR$=\frac{mca+mbx}{a^2}$. And then, in the fimilar triangles NPO, MRN, having NO:PO::NR:MR, the equation will be $ya^4+yca^2q+yxbaq=nmcax+nmbx^2$. And the curve is an hyperbola; and in the cafe of this *Fig.* 4. it will be convex to the plane of the bafe. But when BN is negative in the cafe of *Fig.* 5. the equation, retaining the fame fymbols, will be $ya^4+yqca^2-yxbqa=mncax-nmbx^2$; and the hyperbolic curve will be concave towards the bafe.

6. If the vertical directrix AM is made parallel to the plan of the bafe, but the plane paffing thro' it not parallel to the other directrix, then the fection, meeting the two directrices, will alfo be the fame curve. *Fig.* 6.

For in this cafe the line MR is a conftant quantity; and therefore, if the common fection *a*R of the plane thro' the vertical AM, with the bafe, meet *a*D parallel to NR in *a*, other being as before, making MR$=m$, and D$a=a$; from the analogy NO:PO::NR:MR in the triangles NOP, MRN, we fhall have $mnax=ya^3+yacq+yxqb$; by which the curve is known to be an hyperbola.

And in all thofe fections, where the common interfection of the plane, paffing thro' the vertical with the bafe, is not parallel to the other directrix, the curve is an hyperbola.

7. But now, if we fuppofe the common interfection AR of the plane paffing thro' the vertical, with the bafe, to be parallel to the directrix DN of the bafe, and both directrices to be cut by the fection, the curve will be a parabola. *Fig.* 7.

For

For, in this case, the two lines A R and A Q coincide, and A R is parallel to the directrix D N of the base; and therefore, using the same symbols as above, the equation, from *Fig*. 4. will be reduced to $ya^4 = nmcax + nmbx^2$; and from *Fig*. 5. it will come to $ya^4 = nmcax - nmbx^2$; both which shew the curve to be a parabola.

It may also be demonstrated in this manner. It is evident, that $PO : mr :: PO \times MR : mr \times MR$, that is, the ratio of P O to mr is equal to that compounded of $PO : MR$, and of $MR : mr$. But from the similar triangles M R N, P O N, we have $PO : MR :: NO : NR = nr$, and from the triangles B O N, Brn, we have $BO : Br :: NO : nr$; therefore $PO : MR :: BO : Br$. In like manner, from the equiangular triangles M R A, $m\ r$ A, there will be $MR : mr :: RA : rA$, and from the triangles ROr, Z A R, it is $RA : rA :: Oz : rz$; therefore $MR : mr :: oz : rz$. If then in the ratio of $PO \times RM$ to $mr \times MR$, we substitute the ratio of $BO : Br$, and of $Oz : rz$, which are equal to $PO : MR$, and $MR : mr$, we shall have $PO : mr :: BO \times oz : Br \times rz$; which is a known property of the parabola.

And thus I have endeavoured to extend, a little, the Theory of the conic sections. I have here shewn how two of them may be had from the sections of this solid; and in the year 1733, I published, in my *Exercitatio Geometrica*, a method of describing all of them on a plane, by the moving of three right lines about three given points, two of the intersections being drawn along two other lines given in position; the only hint of which I had from a geometric

tric locus, in the construction of a clock, by the celebrated M. De la Hire, in the French *Memoires* for the year 1717, intituled, *Construction d'une Horologe qui marque le vrai tems avec le moyen.*

Having now briefly considered the various sections of this solid, the directrix DN of the base being a right line, let us next see what they are when it is a curve line of any order. But because there are an infinite number of cases, it will be sufficient to mention a few of them, when that directrix is a conic section or circle, and then to give a general proposition when it is any higher geometrical curve.

8. If the directrix DN of the base be a parabola, having a diameter parallel to the intersection AR of the plane, passing thro' the vertical A, and other things being as before, the section BP will be a line of the third order. *Fig.* 8.

Let the diameter be D L n, the ordinate N n, and the equation of the parabola $u^2 = zp$, and make the moving plan MNR to pass thro' the ordinate N n, and the section to intersect the base in BO, and the diameter in E. From the points B and D draw BL and DC, parallel to NO, and meeting the lines AR and D n in C and L. Then suppose AR : RM : : $a:m$, and BO : L n : : $a:n$, AC $=c$, DC $=d$, BL $=b$, DL $=l$, BE $=e$, N $n=u$, D $n=z$, the abscisse BO $=x$, the ordinate PO $=y$; and we have L $n=z - l = \frac{nx}{a}$, $no = \frac{b \times x - e}{e}$, MR $= \frac{mc + mz}{a} = \frac{mca + mal + nxm}{a^2}$ and $u^2 = \frac{apl + pxn}{a}$. But from the similar triangles NPO and NMR we have this proportion NO : PO : : NR : MR, which gives the

the equation $\frac{\overline{bx - be} \times \overline{mca + mal + nmx} - dya^2e}{ya^2e - mcae - male - nmxe} = u = \sqrt{\frac{pl + pxn}{a}}$; and this being reduced, ſhews, that the ſection P B is a line of the third order.

9. If the directrix D N be a circle, and other things being as before, the ſection will be a line of the fourth order. *Fig.* 9.

Make the center of the circle to be in the line D L n parallel to A Q, the ordinate to be N $n = u$, the abſciſſe D $n = z$, the radius $= r$, and the equation $u^2 = 2rz - z^2$. And let the plane paſſing thro' the vertical, cut the baſe in A R, and the ſection meet the diameter D K in E. Then the ſame things being ſuppoſed, and the ſymbols retained as before, we ſhall have L $n = z - l = \frac{xn}{a}$, $z = \frac{al + xn}{a}$, n O $= \frac{bx - eb}{e}$, Q R $= \frac{qac + qal + qnx}{a^2}$, M R $= \frac{mac + mal + mnx}{a^2}$, $\sqrt{\frac{2rla^2 + 2rxna - x^2n^2 - 2alnx - a^2l^2}{a^2}} = u$. And the analogy N O : P O : : N R : M R will give this equation, $\frac{mac + mal + mnx}{a^2} \times \frac{be - bx}{e} + \frac{ya^2d + yqac + yqal + yqnx}{a^2} = \frac{mac + mal + mnx - a^2y}{a^2} \times \sqrt{\frac{2rla^2 + 2rxna - x^2n^2 - 2alnx - a^2l^2}{a^2}}$; from which it appears, that the curve B P is a line of the fourth order.

And in general it may be ſeen, that if the directrix of the baſe be a conic ſection, except in the caſe above, the ſection of the ſolid will be a line of the fourth order.

10.

10. If the vertical be parallel to the baſe, and the plane paſſing thro' it perpendicular, the directrix of the baſe being a circle, having its center in the interſection A R of the two planes; then the ſolid will be the cono-cuneus of the learned Dr. Wallis, and the various curve ſections of it will be alſo lines of the fourth order. *Fig.* 10.

In this caſe, the quantities Q R and D C will vaniſh, and making M R $= m$, the equation, retaining the other ſymbols, will be $\overline{m - y} \times \sqrt{\frac{2rxna + 2rla^2 - x^2 - 2xnla - a^2l^2}{a^2}} = m \times \frac{eb - xb}{e}$. And here it is ſurpriſing that the great Doctor, while he was conſidering his ſolid, did not fall upon the one I have explained; but indeed, in ſearching after new diſcoveries, we are often like thoſe, who, groping in the dark, miſs the things that are neareſt them.

11. To conclude, if the directrix D N of the baſe be a line of any order n, the ſection B P will be of the order $2n$. *Fig.* 11.

In the equation of the curve directrix D N of the baſe $u^n = Az^n + Bz^{n-1}u + Cz^{n-2}u^2 + Dz^{n-3}u^3$ &c. make the abſciſſe D $n = z$, and the ordinate N $n = u$; and draw A Q parallel to D n, and then, other things being as before, the analogy N O : P O :: N R : M R will be thus expreſſed, $u + \frac{bx - eb}{e} : y :: u + d + \frac{qnx + qal + qac}{a^2} : \frac{nmx + mal + mca}{a^2}$; from which we ſhall have $u =$

$$\frac{\overline{eb - bx} \times \overline{nmx + mca + mal}}{emal + emnx + mace - a^2ey} + \frac{da^2y + nqxy + qaly + qyac}{mnx + mac - a^2y + mal}.$$

And becauſe D $n = z = l + \frac{nx}{a}$, if we ſubſtitute theſe values of u and z, in the above general equation,

the line of the ſection B P will appear to be of the order 2 *n*.

And now, my Lord, I have given you ſome general propoſitions about the various ſections of this ſolid, and I have ſhewn how lines of any order may ariſe from them; which is a new theory, and perhaps may introduce to ſomething farther. I have other things of this ſort, that relate to what I have publiſhed in the Philoſophical Tranſactions, Vol. 39. in the year 1735, which I have had many years by me, that I intend to ſend you, if my health, and the circumſtances of my life, will allow me to reviſe them. And in the mean time, with great reſpect, I am,

My Lord,

Your Lordſhip's moſt obedient

and affectionate ſervant,

Sion College, Dec. 18, 1759.

Wm. Brakenridge.

END *of* PART I.